In the Name of God

DYNAMICS

[Course No. 8102128]

Dr. Mehdi Ghassemieh

m.ghassemieh@ut.ac.ir
Tel. 6111-2273
Fax. 6640-3808

پردیس دانشکده های فنی
University College of Engineering

بنام خدا

دینامیک (نیمسال ۲-۹۷-۹۶)

شماره درس ۸۱۰۲۱۲۸

دکتر مهدی قاسمیه

دانشکده مهندسی عمران

فصل نهم :

Mechanical Vibration

ارتعاشات مکانیکی

ارتعاش آزاد ذره یا نقطه مادی (بدون نیروی خارجی) **Free vibration**

ارتعاش آزاد نا میرا

ارتعاش آزاد آونگ ساده

ارتعاش آزاد جسم صلب

اصل حفظ انرژی مکانیکی **Conservation of energy**

ارتعاش زوری (با نیروی خارجی) **Forced vibration**

ارتعاش آزاد با میرائی **Damped free vibration**

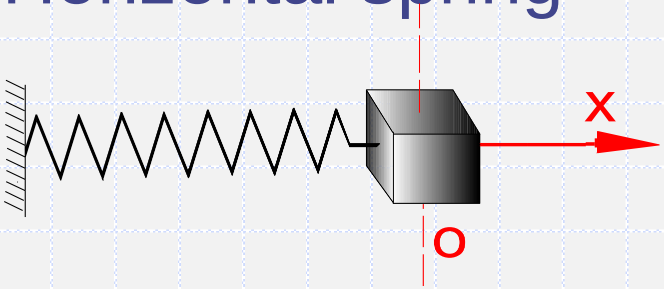
ارتعاش زوری با میرائی **Damped forced vibration**

Free Vibrations of Particles (Simple Harmonic Motion)

ارتعاش آزاد ذره یا نقطه مادی (بدون نیروی خارجی)

$$f = -kx$$

Mass on a Horizontal spring



$$m\ddot{x} = -kx$$

$$\ddot{x} + \frac{k}{m}x = 0$$

Differential Equation

$$\ddot{x} + \omega_n^{\,2} x = 0$$

ω_n is the *natural circular frequency*

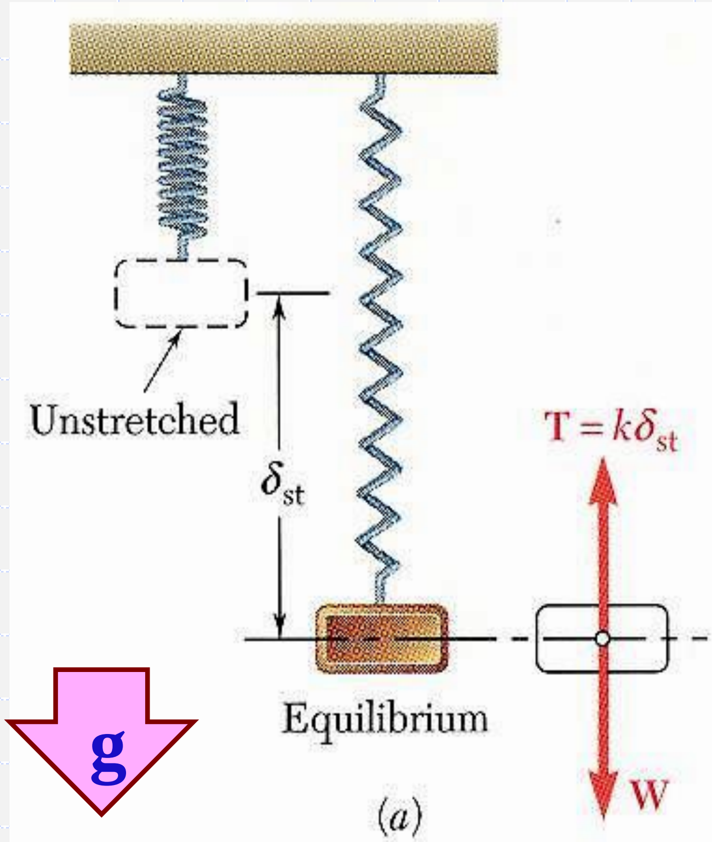
$$x = A\sin\left(\sqrt{\frac{k}{m}}\,t\right) + B\cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

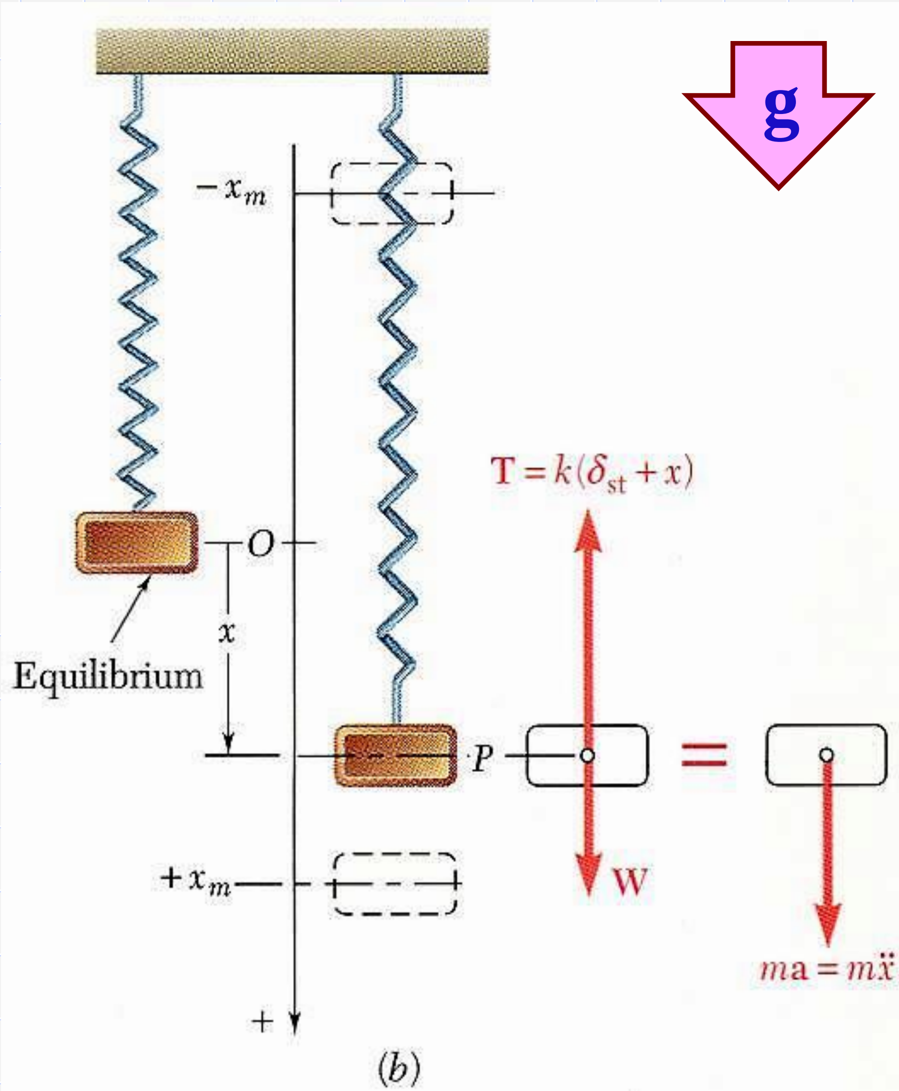
$$A = v_0/\omega_n$$

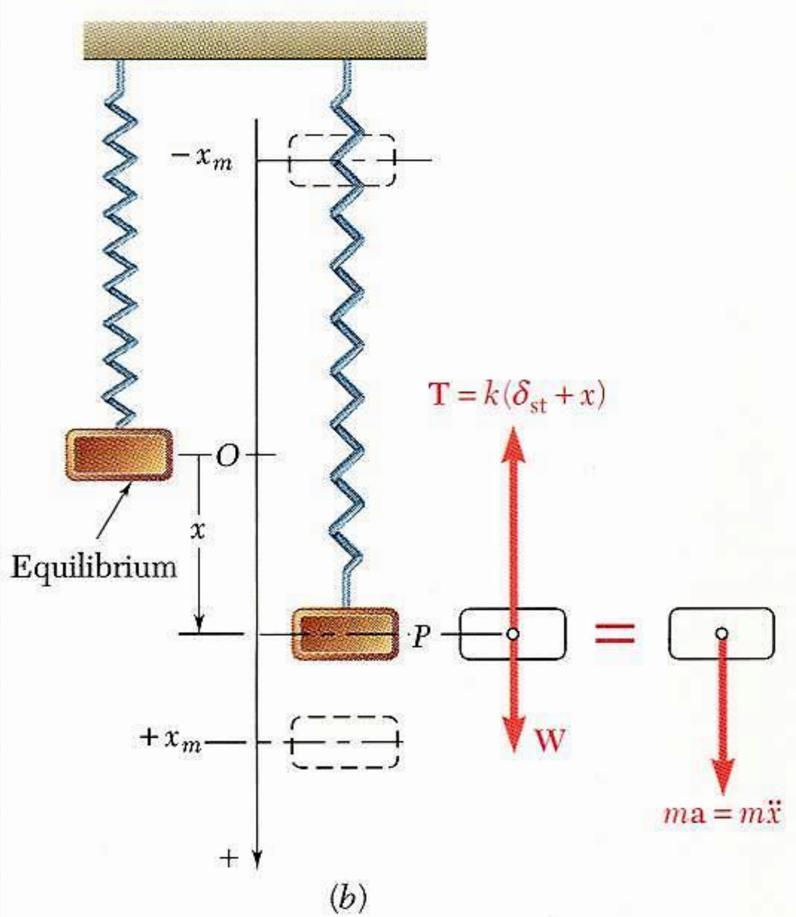
$$B = x_0$$

Mass on a Vertical spring



$W = T = k\delta_{st}$





- If a particle is displaced through a distance x_m from its equilibrium position and released with no velocity, the particle will undergo *simple harmonic motion*,

$$ma = F = W - k(\delta_{st} + x) = -kx$$
$$m\ddot{x} + kx = 0$$

- General solution is

$$x = A\sin\left(\sqrt{\frac{k}{m}}\,t\right) + B\cos\left(\sqrt{\frac{k}{m}}\,t\right)$$
$$= A\sin(\omega_n t) + B\cos(\omega_n t)$$

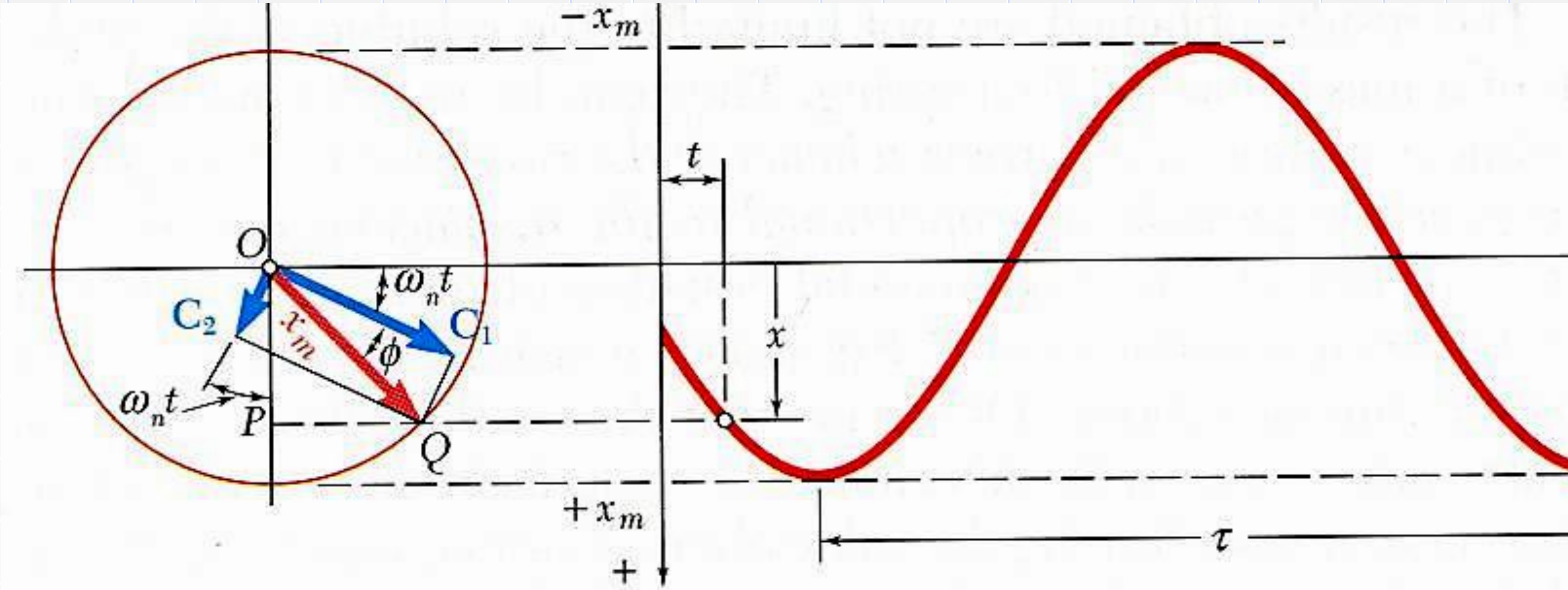
  - *x* is a *periodic function* and ω_n is the *natural circular frequency* of the motion.

$$\omega_n = \sqrt{\frac{k}{m}}$$

- *A* and *B* are determined by the initial conditions:

$$x = A\sin(\omega_n t) + B\cos(\omega_n t) \qquad B = x_0$$

$$v = \dot{x} = A\omega_n\cos(\omega_n t) - B\omega_n\sin(\omega_n t) \qquad A = v_0/\omega_n$$



$$A = \frac{v_0}{\omega_n}$$
$$B = x_0$$

- Displacement is equivalent to the *x* component of the sum of two vectors $\vec{C}_1 + \vec{C}_2 = \vec{A} + \vec{B}$ which rotate with constant angular velocity ω_n.

$$x = x_m \sin(\omega_n t + \phi)$$

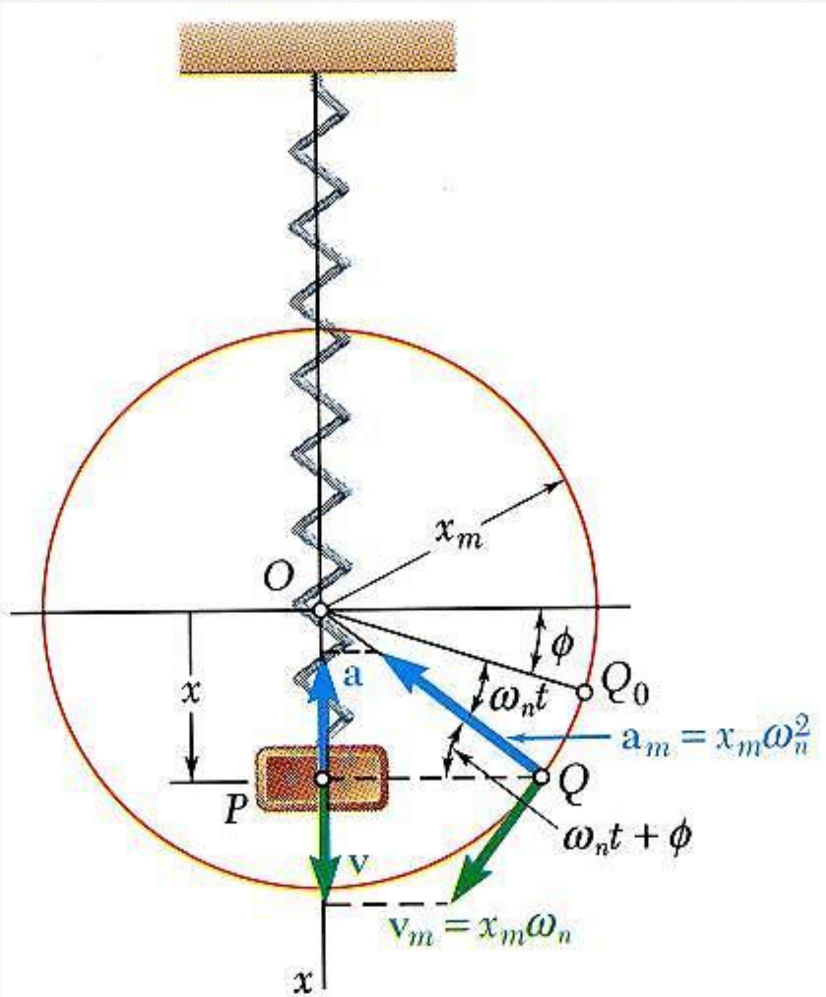
$$x_m = \sqrt{(v_0/\omega_n)^2 + x_0^2} =$$ ***amplitude***

$$\phi = \tan^{-1}(v_0/x_0\omega_n) =$$ ***phase angle***

$$\tau_n = \frac{2\pi}{\omega_n} =$$ ***period***

$$f_n = \frac{1}{\tau_n} = \frac{\omega_n}{2\pi} =$$ ***natural frequency***

- **Velocity-time and acceleration-time curves can be represented by sine curves of the same period as the displacement-time curve but different phase angles.**

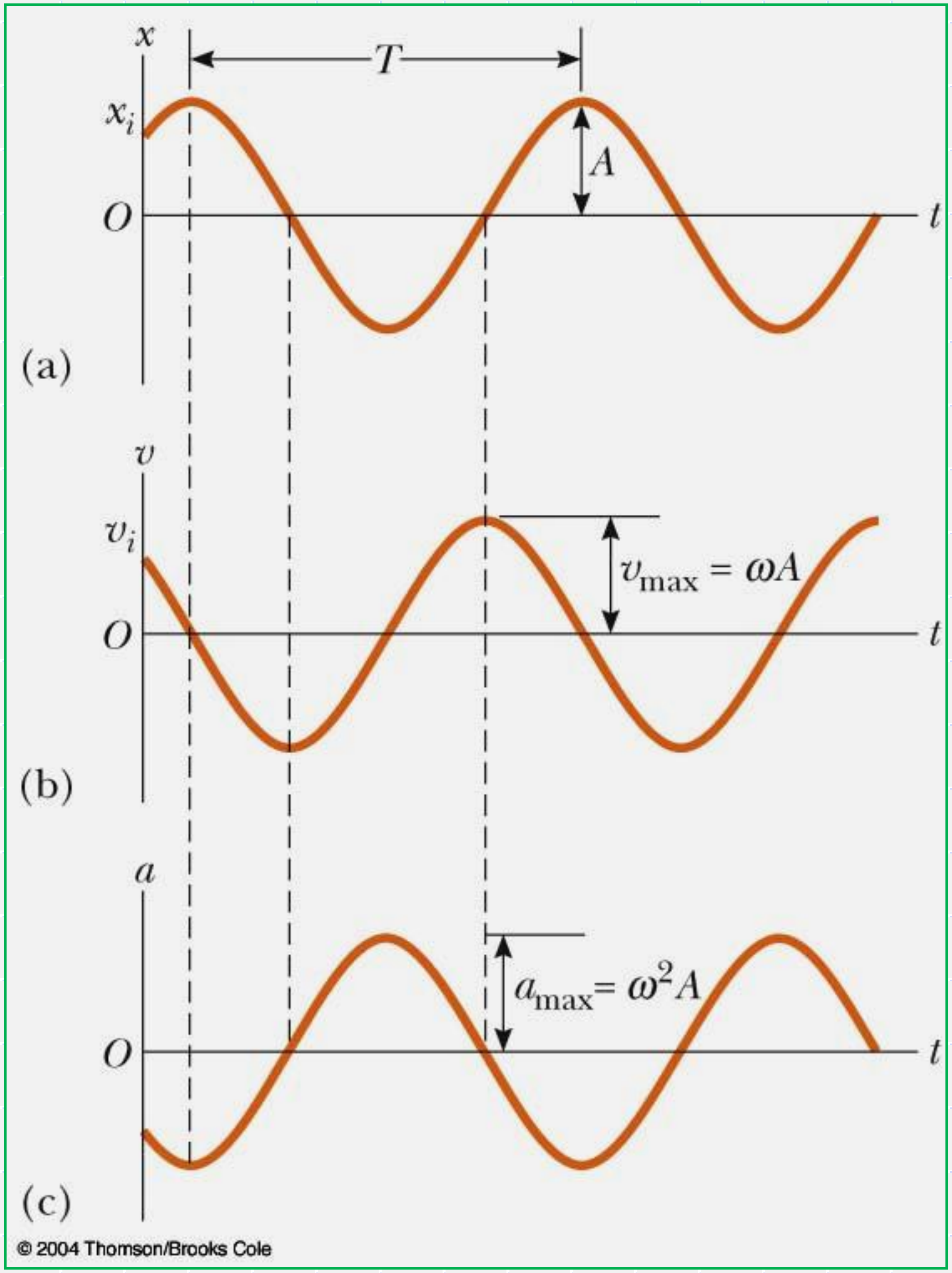


$$x = x_m \sin(\omega_n t + \phi)$$

$$v = \dot{x}$$
$$= x_m \omega_n \cos(\omega_n t + \phi)$$
$$= x_m \omega_n \sin(\omega_n t + \phi + \pi/2)$$

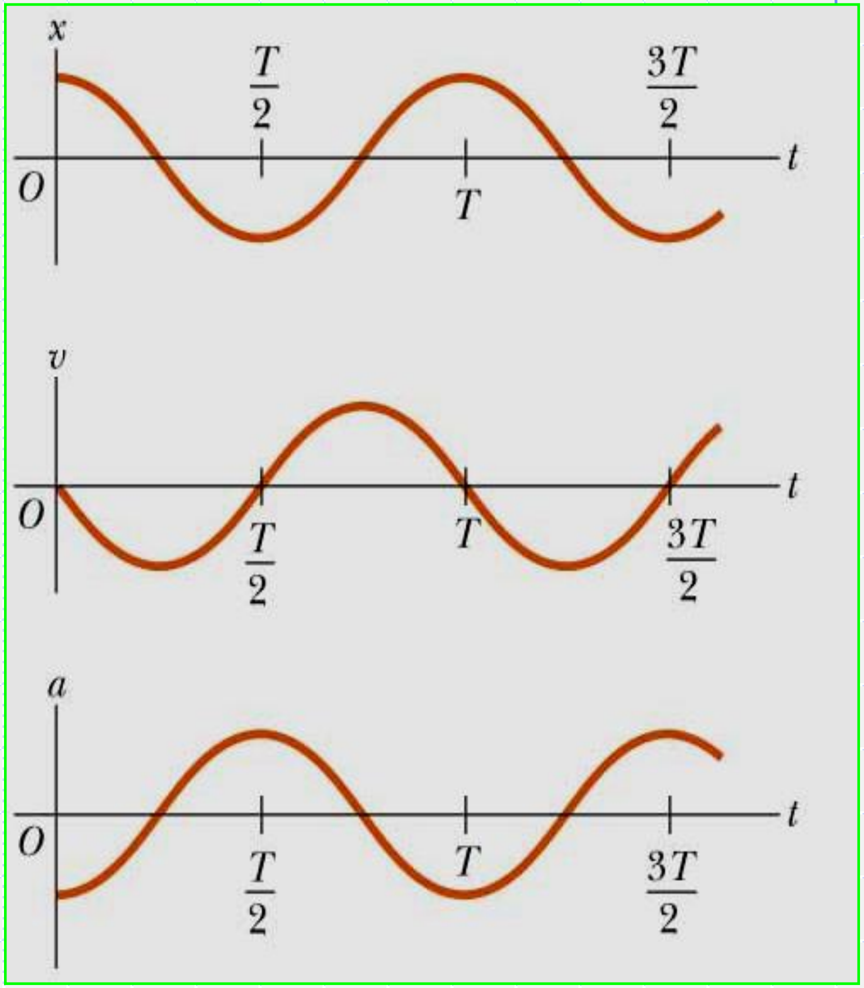
$$a = \ddot{x}$$
$$= -x_m \omega_n^2 \sin(\omega_n t + \phi)$$
$$= x_m \omega_n^2 \sin(\omega_n t + \phi + \pi)$$

- The graphs show:
  - (a) displacement as a function of time
  - (b) velocity as a function of time
  - (c) acceleration as a function of time
- The velocity is 90° out of phase with the displacement and the acceleration is 180° out of phase with the displacement



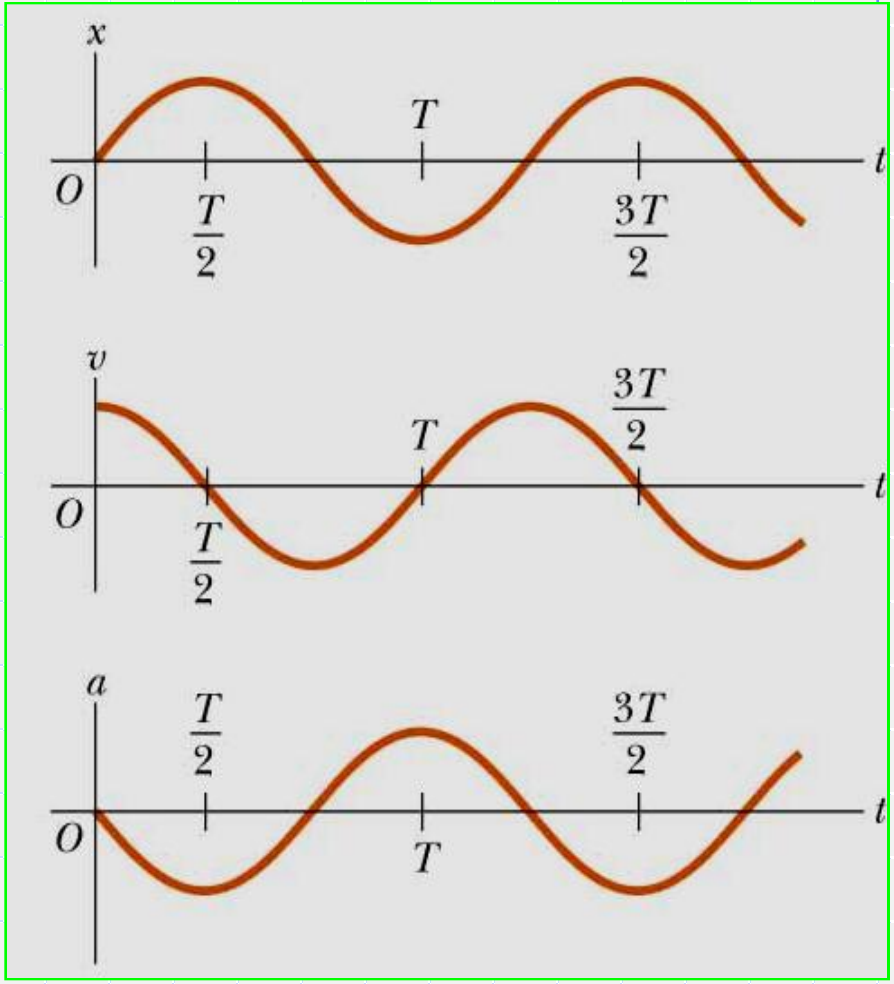
SHM Example 1

- Initial conditions at $t = 0$ are
  - $x(0) = A$
  - $v(0) = 0$
- This means $\phi = 0$
- The acceleration reaches extremes of $\pm \omega^2 A$
- The velocity reaches extremes of $\pm \omega A$

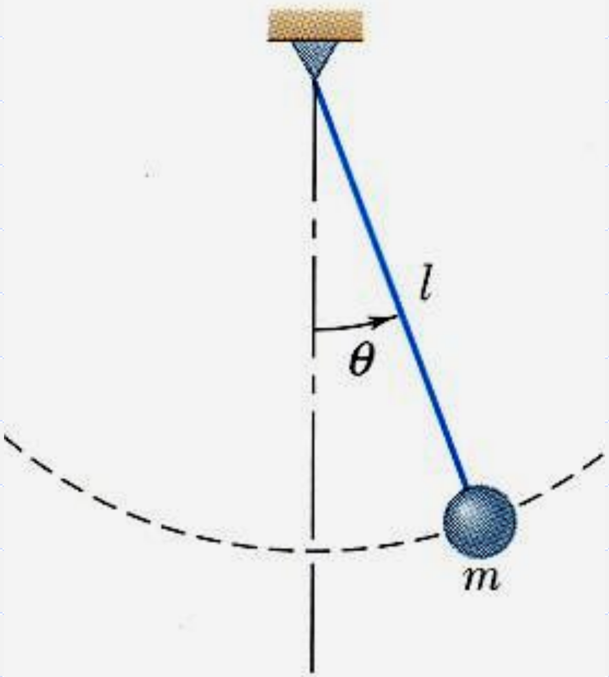


SHM Example 2

- Initial conditions at $t = 0$ are
  - $x(0)=0$
  - $v(0) = v_i$
- This means $\phi = -\pi/2$
- The graph is shifted one-quarter cycle to the right compared to the graph of $x(0) = A$



Simple Pendulum (Approximate Solution) آونگ ساده



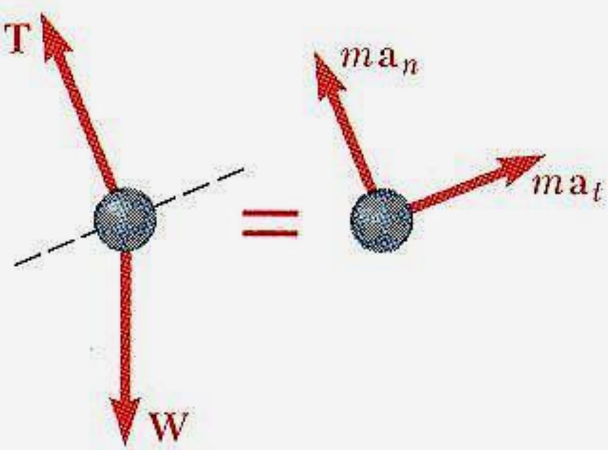
- Results obtained for the spring-mass system can be applied whenever the resultant force on a particle is proportional to the displacement and directed towards the equilibrium position.

- Consider tangential components of acceleration and force for a simple pendulum,

$$\sum F_t = m\,a_t$$

$$-W\sin\theta = ml\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$



for small angles,

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\theta = \theta_m \sin(\omega_n t + \phi)$$

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}}$$

$$\omega_n = \sqrt{\frac{g}{\ell}}$$

Simple Pendulum (Exact Solution)

An exact solution for

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

leads to

$$\tau_n = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\sin^2(\theta_m/2)\sin^2\phi}}$$

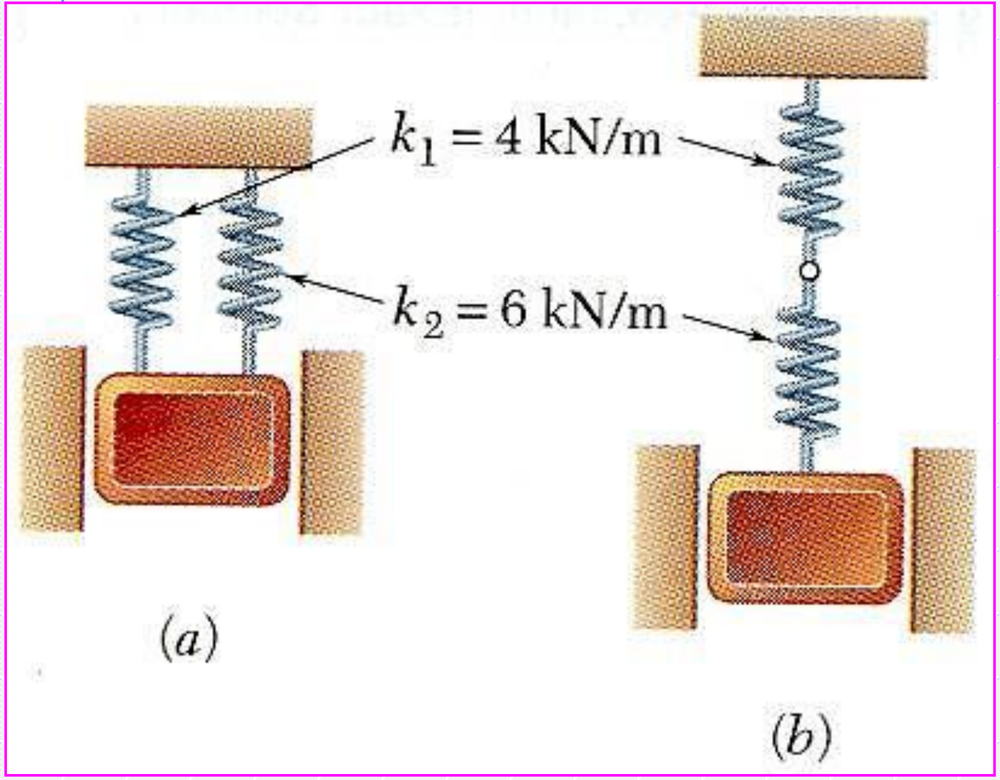
which requires numerical solution.

$$\tau_n = \frac{2K}{\pi}\left(2\pi\sqrt{\frac{l}{g}}\right)$$

Table 19.1. Correction Factor for the Period of a Simple Pendulum

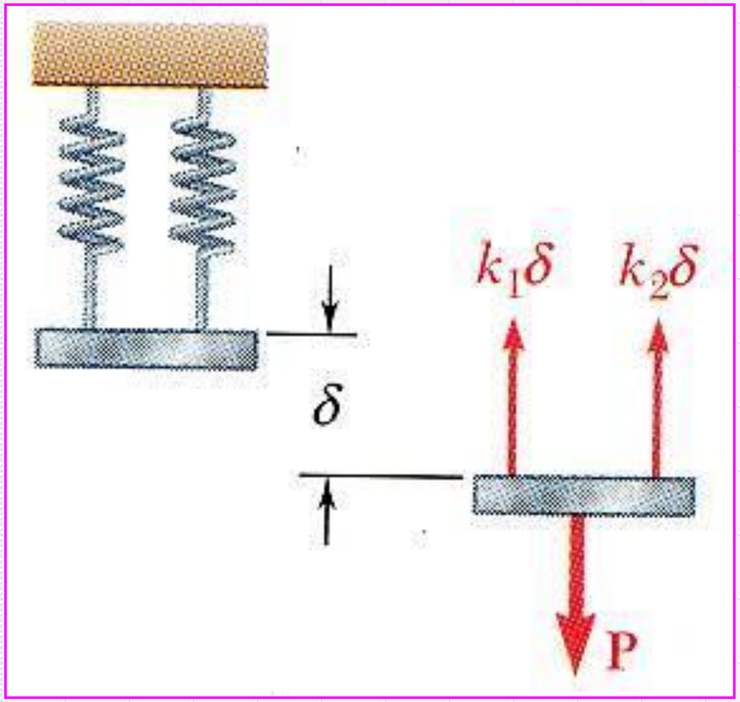
θ_m	0°	10°	20°	30°	60°	90°	120°	150°	180°
K	1.571	1.574	1.583	1.598	1.686	1.854	2.157	2.768	∞
$2K/\pi$	1.000	1.002	1.008	1.017	1.073	1.180	1.373	1.762	∞

مثال: بلوک 50 کیلوگرمی در دو سیستم متفاوت به دو فنر متصل گردیده است. اگر بلوک به اندازه 40 میلیمتر از حالت تعادل به سمت پائین کشیده و رها گردد: مطلوبست : الف) فرکانس هر یک از سیستم ها ، ب) حداکثر سرعت بلوک و ج) حداکثر شتاب بلوک



- For each spring arrangement, determine the spring constant for a single equivalent spring.
- Apply the approximate relations for the harmonic motion of a spring-mass system.

$$k_1 = 4 \ \text{kN/m} \qquad k_2 = 6 \ \text{kN/m}$$



$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10^4}{20}} = 14.14 \,\text{rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n} = 0.444 \ (\text{sec.})$$

$$P = k_1\delta + k_2\delta$$

$$k = \frac{P}{\delta} = k_1 + k_2$$

$$= 10 \ \text{kN/m} = 10^4 \ \text{N/m}$$

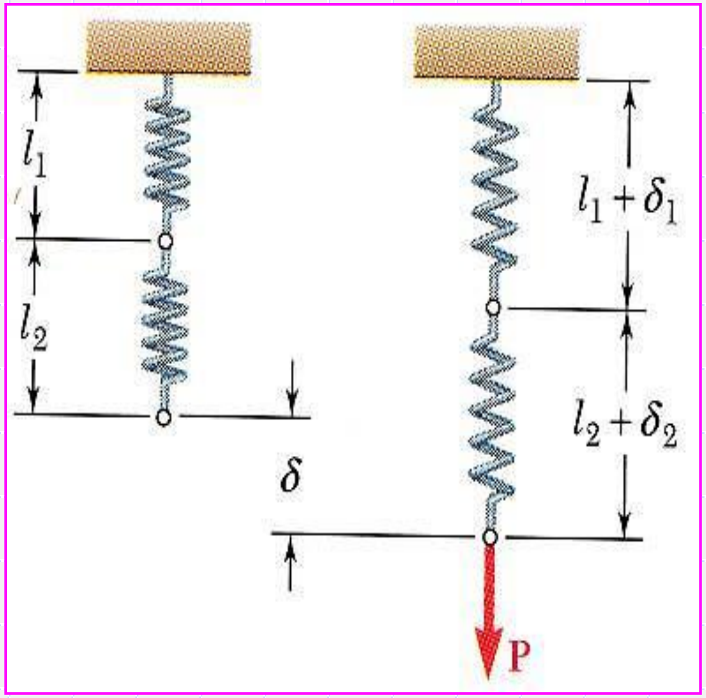
$$v_m = x_m \omega_n$$

$$= (0.040)(14.14) = 0.566 \ (m/s)$$

$$a_m = x_m a_n^2$$

$$= (0.040)(14.14)^2 = 8 \ (m/s^2)$$

$$k_1 = 4 \ \text{kN/m} \qquad k_2 = 6 \ \text{kN/m}$$



$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2400\text{N/m}}{20\,\text{kg}}} = 6.93\,\text{rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n} = 0.907 \ \ (\text{sec.})$$

$$\delta = \delta_1 + \delta_2 = \frac{P}{k_1} + \frac{P}{k_2}$$

$$k = \frac{P}{\delta} = \frac{P}{\dfrac{P}{k_1} + \dfrac{P}{k_2}} = \frac{k_1 k_2}{k_1 + k_2}$$

$$= 2400 \ \text{N/m}$$

$$v_m = x_m \omega_n$$

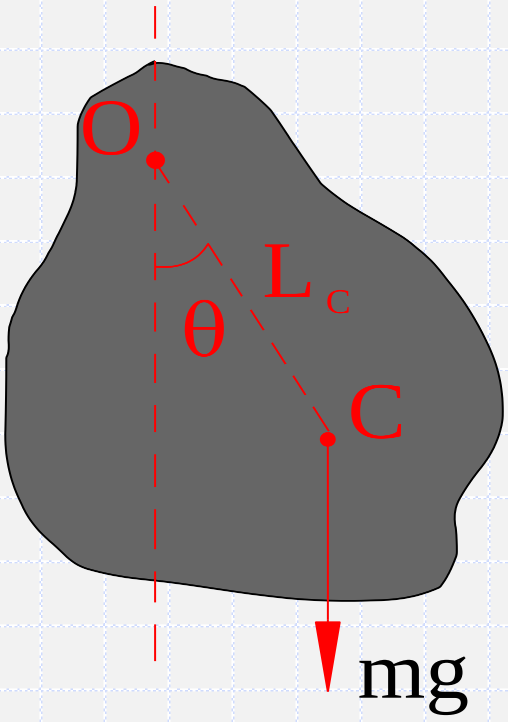
$$= (0.040)(6.93) = 0.277 \ \ (m/s)$$

$$a_m = x_m a_n^2$$

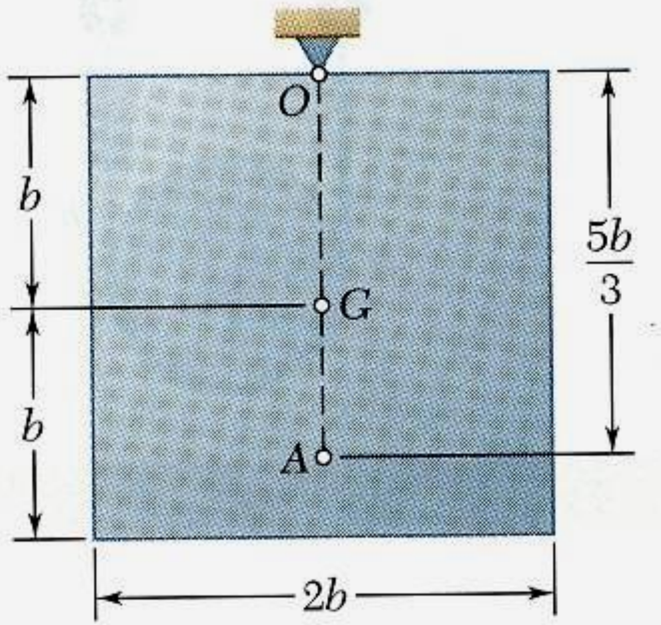
$$= (0.040)(6.93)^2 = 1.92 \ \ (m/s^2)$$

Free Vibrations of Rigid Body ارتعاش آزاد جسم صلب

- $$-mg\sin\theta l_c = I_o\alpha$$
- for small angles $\sin\theta \approx \theta$
- $$-mg\theta l_c = I_o\ddot{\theta}$$
- $$\ddot{\theta} + \frac{mgl_c}{I_o}\theta = 0$$
- $$\ddot{\boldsymbol{\theta}} + \boldsymbol{\omega}^2\boldsymbol{\theta} = 0$$



Free Vibrations of Rigid Body ارتعاش آزاد جسم صلب



- If an equation of motion takes the form

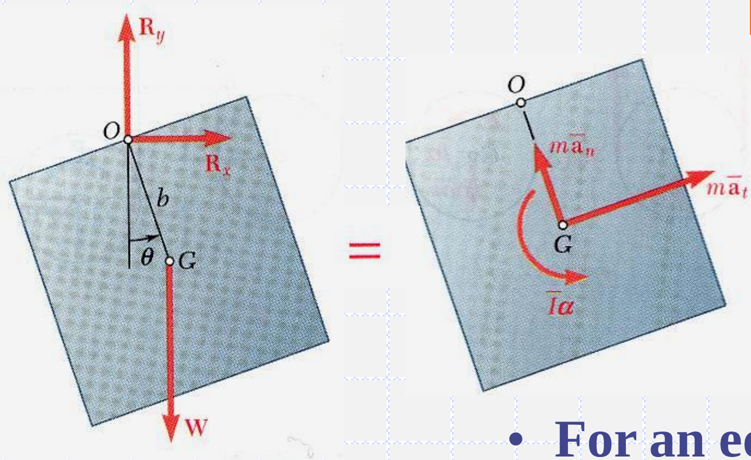
$$\ddot{x} + \omega_n^2 x = 0 \quad \text{or} \quad \ddot{\theta} + \omega_n^2 \theta = 0$$

- the corresponding motion may be considered as simple harmonic motion.
- Analysis objective is to determine ω_n.
- Consider the oscillations of a square plate

$$+\circlearrowleft \; -W(b\sin\theta) = (mb\ddot{\theta}) + \bar{I}\ddot{\theta}$$

$$\text{but } \bar{I} = \tfrac{1}{12} m\left[(2b)^2 + (2b)^2\right] = \tfrac{2}{3} mb^2, \quad W = mg$$

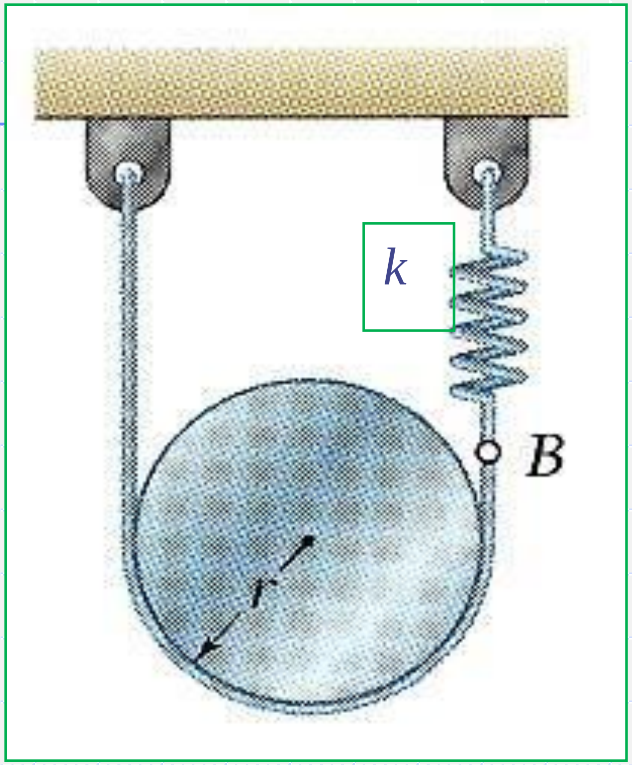
$$\ddot{\theta} + \frac{3}{5}\frac{g}{b}\sin\theta \cong \ddot{\theta} + \frac{3}{5}\frac{g}{b}\theta = 0$$



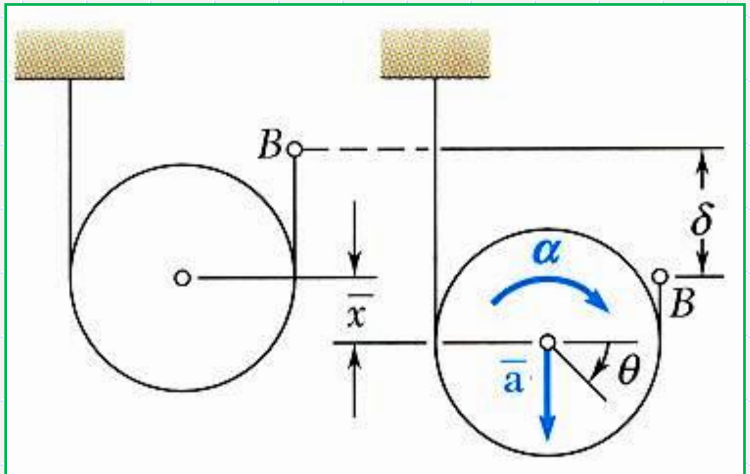
$$\text{then } \omega_n = \sqrt{\frac{3g}{5b}}, \quad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{5b}{3g}}$$

- **For an equivalent simple pendulum,** $l = 5b/3$

مثال: دیسک با وزن **W** به حالت نشان داده شده معلق نگهداشته شده است که رها میگردد. مطلوبست : فرکانس طبیعی دیسک.

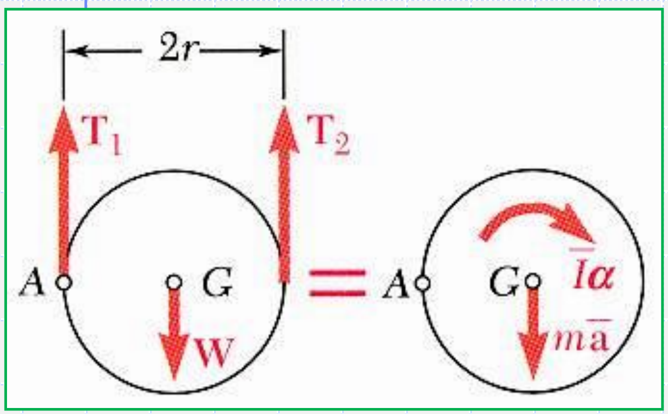


- **From the kinematics of the system, relate the linear displacement and acceleration to the rotation of the cylinder.**
- **Based on a free-body-diagram equation for the equivalence of the external and effective forces, write the equation of motion.**
- **Substitute the kinematic relations to arrive at an equation involving only the angular displacement and acceleration.**



$$\bar{x} = r\theta \qquad \delta = 2\bar{x} = 2r\theta$$

$$\vec{\alpha} = \ddot{\theta} \; +\!\curvearrowright \qquad \bar{a} = r\alpha = r\ddot{\theta} \qquad \vec{\bar{a}} = r\ddot{\theta} \; +\!\downarrow$$



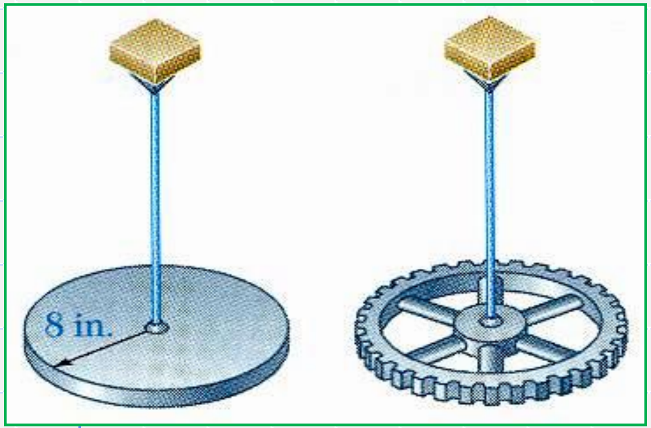
$$+\!\curvearrowright \sum M_A = \sum (M_A)_{eff} : \qquad Wr - T_2(2r) = m\bar{a}r + \bar{I}\alpha$$

$$\text{but} \quad T_2 = T_0 + k\delta = \tfrac{1}{2}W + k(2r\theta)$$

$$Wr - \left(\frac{1}{2}W + 2kr\theta\right)(2r) = m\left(r\ddot{\theta}\right) r + \frac{1}{2} mr^2 \theta$$

$$\ddot{\theta} + \frac{8}{3}\frac{k}{m}\theta = 0$$

$$\omega_n = \sqrt{\frac{8k}{3m}} \qquad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{3m}{8k}} \qquad f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{8k}{3m}}$$



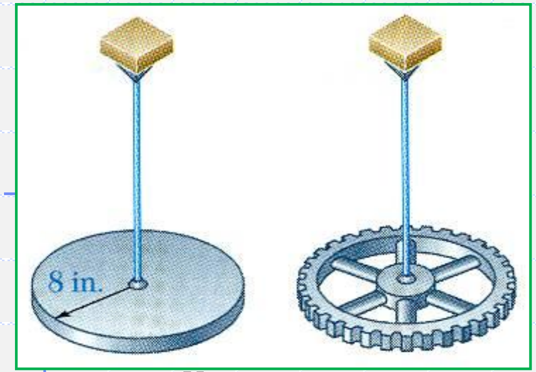
$W = 20\ \text{lb}$

$\tau_n = 1.13\,\text{s}$ $\tau_n = 1.93\,\text{s}$

The disk and gear undergo torsional vibration with the periods shown. Assume that the moment exerted by the wire is proportional to the twist angle.

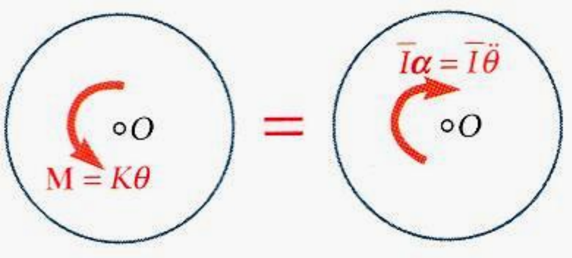
Determine *a*) the wire torsional spring constant, *b*) the centroidal moment of inertia of the gear, and *c*) the maximum angular velocity of the gear if rotated through 90º and released.

- Using the free-body-diagram equation for the equivalence of the external and effective moments, write the equation of motion for the disk/gear and wire.
- With the natural frequency and moment of inertia for the disk known, calculate the torsional spring constant.
- With natural frequency and spring constant known, calculate the moment of inertia for the gear.
- Apply the relations for simple harmonic motion to calculate the maximum gear velocity.



$W = 20\text{ lb}$

$\tau_n = 1.13\text{s}$ $\qquad \tau_n = 1.93\text{ s}$



SOLUTION:

- Using the free-body-diagram equation for the equivalence of the external and effective moments, write the equation of motion for the disk/gear and wire.

$$+\circlearrowleft \sum M_O = \sum (M_O)_{eff} :$$

$$+K\theta = -\bar{I}\,\ddot{\theta}$$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$$

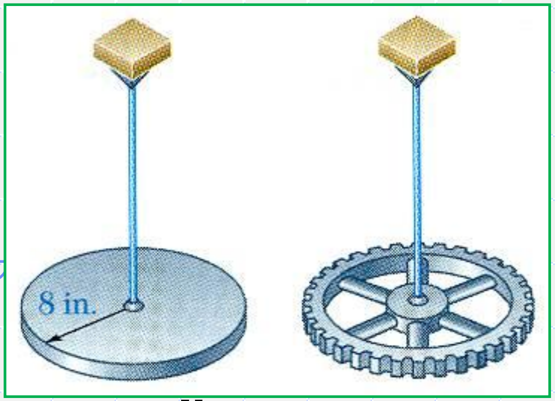
$$\omega_n = \sqrt{\frac{K}{\bar{I}}} \qquad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\bar{I}}{K}}$$

- With the natural frequency and moment of inertia for the disk known, calculate the torsional spring constant.

$$\bar{I} = \tfrac{1}{2}mr^2 = \frac{1}{2}\left(\frac{20}{32.2}\right)\left(\frac{8}{12}\right)^2 = 0.138\,\text{lb}\cdot\text{ft}\cdot\text{s}^2$$

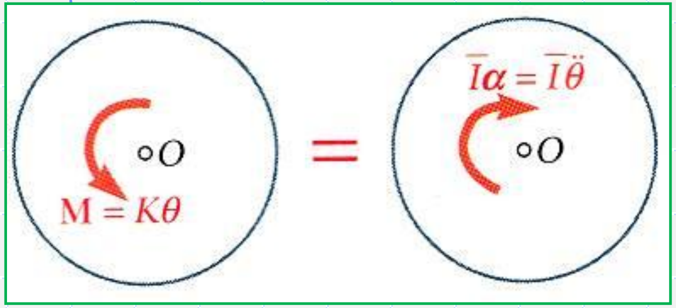
$$1.13 = 2\pi\sqrt{\frac{0.138}{K}}$$

$$K = 4.27\ \text{lb}\cdot\text{ft/rad}$$



$W = 20\text{ lb}$

$\tau_n = 1.13\text{s}$ $\qquad \tau_n = 1.93\text{ s}$



- With natural frequency and spring constant known, calculate the moment of inertia for the gear.

$$1.93 = 2\pi\sqrt{\frac{\bar{I}}{4.27}}$$

$$\bar{I} = 0.403\,\text{lb}\cdot\text{ft}\cdot\text{s}^2$$

- Apply the relations for simple harmonic motion to calculate the maximum gear velocity.

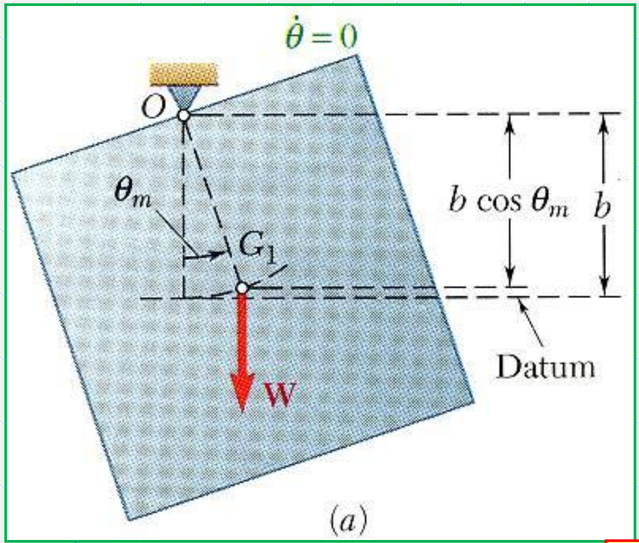
$$\theta = \theta_m \sin\omega_n t \qquad \omega = \theta_m\omega_n \sin\omega_n t \qquad \omega_m = \theta_m\omega_n$$

$$\theta_m = 90° = 1.571\,\text{rad}$$

$$\omega_m = \theta_m\left(\frac{2\pi}{\tau_n}\right) = (1.571\,\text{rad})\left(\frac{2\pi}{1.93\,\text{s}}\right)$$

$$\omega_m = 5.11\text{rad/s}$$

Principle of Conservation of Energy اصل حفظ انرژی مکانیکی



$$T + V = const.$$

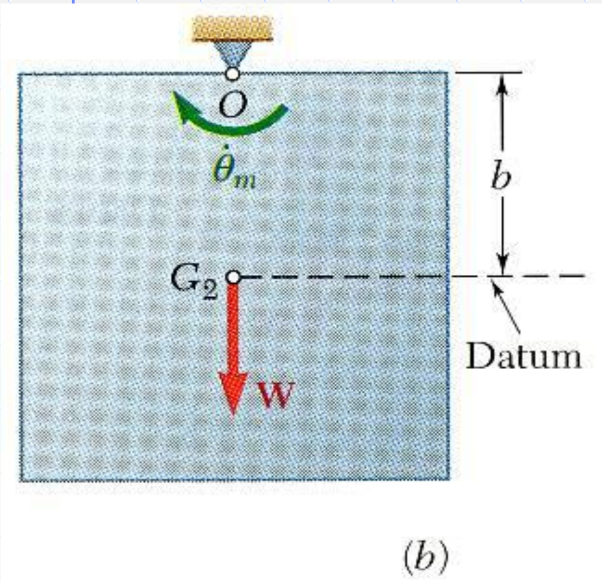
$$\tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} kx^2 = const.$$

$$\dot{x}^2 + \omega_n^2 x^2 =$$

$$T_1 = 0$$

$$V_1 = Wb(1-\cos\theta) = Wb\left[2\sin^2(\theta_m/2)\right]$$

$$\cong \tfrac{1}{2} Wb\theta_m^2$$



$$T_2 = \tfrac{1}{2} m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\,\omega_m^2$$

$$= \tfrac{1}{2} m\left(b\dot\theta_m\right)^2 + \tfrac{1}{2}\left(\tfrac{2}{3} mb^2\right)\omega_m^2$$

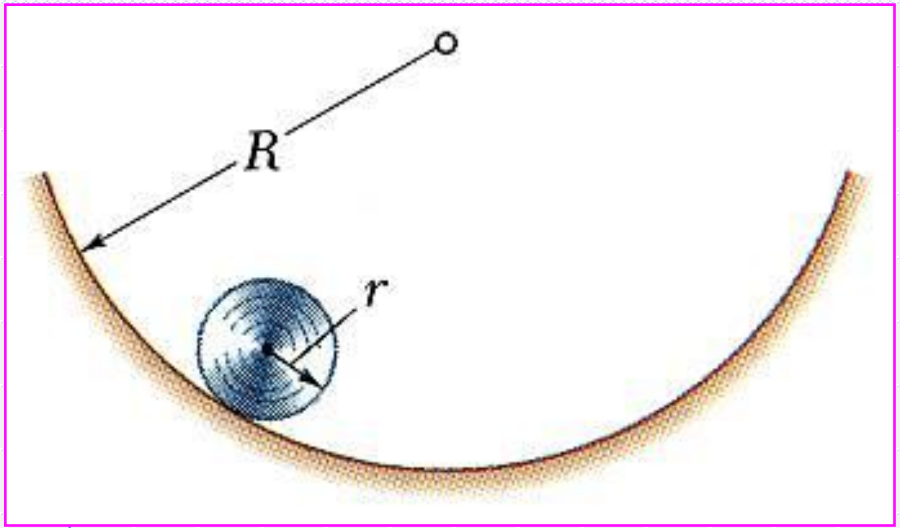
$$= \tfrac{1}{2}\left(\tfrac{5}{3} mb^2\right)\dot\theta_m^2$$

$$V_2 = 0$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 + \tfrac{1}{2} Wb\theta_m^2 = \tfrac{1}{2}\left(\tfrac{5}{3} mb^2\right)\theta_m^2\omega_n^2 + 0$$

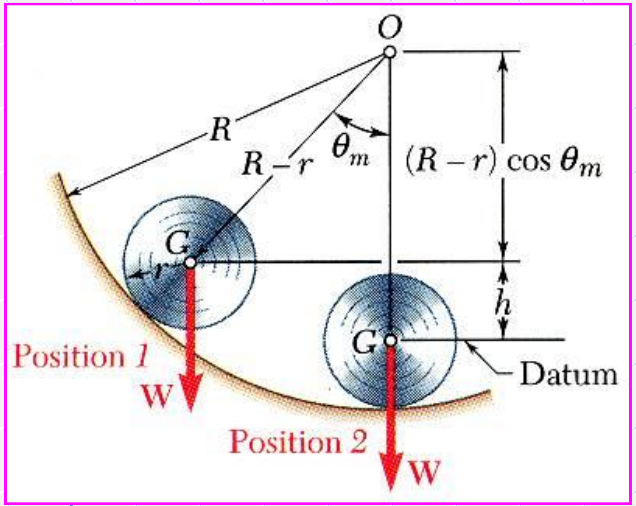
$$\omega_n = \sqrt{3g/5b}$$



Determine the period of small oscillations of a cylinder which rolls without slipping inside a curved surface.

SOLUTION:

- Apply the principle of conservation of energy between the positions of maximum and minimum potential energy.
- Solve the energy equation for the natural frequency of the oscillations.



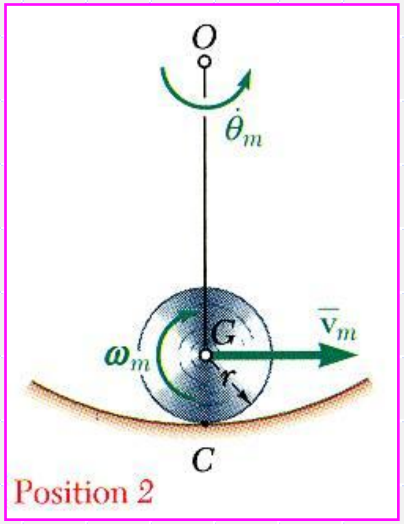
SOLUTION:

- Apply the principle of conservation of energy between the positions of maximum and minimum potential energy.

$$T_1 + V_1 = T_2 + V_2$$

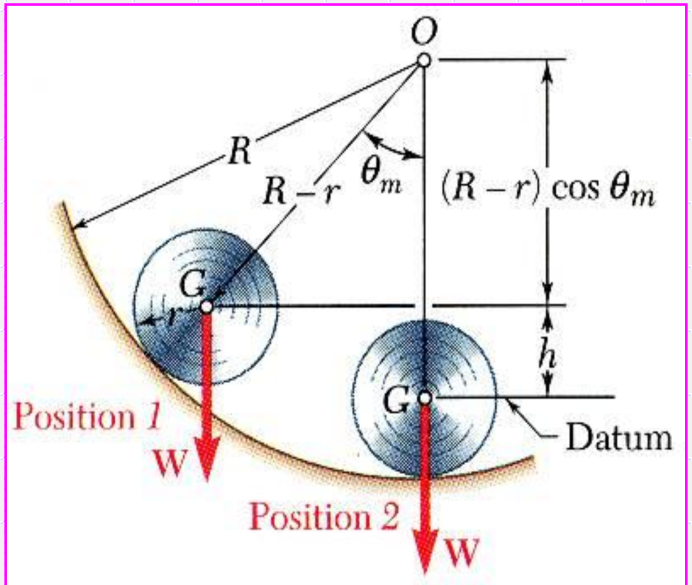
$$T_1 = 0$$

$$V_1 = Wh = W(R-r)(1-\cos\theta) \cong W(R-r)\left(\theta_m^2/2\right)$$



$$T_2 = \tfrac{1}{2} m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\omega_m^2 = \tfrac{1}{2} m(R-r)\dot{\theta}_m^2 + \tfrac{1}{2}\left(\tfrac{1}{2} mr^2\right)\left(\frac{R-r}{r}\right)^2 \dot{\theta}_m^2 = \tfrac{3}{4} m(R-r)^2 \dot{\theta}_m^2$$

$$V_2 = 0$$



- Solve the energy equation for the natural frequency of the oscillations.

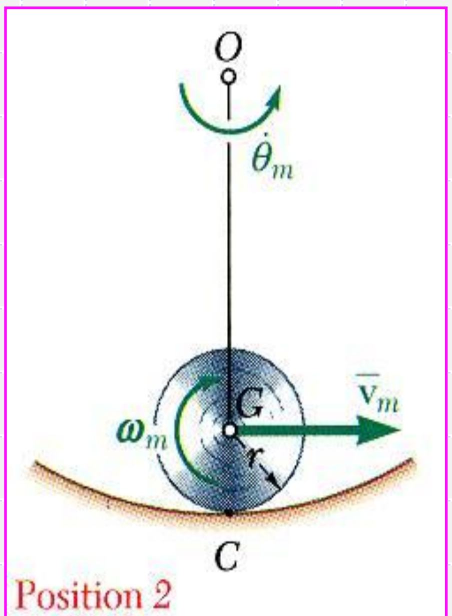
$$T_1 + V_1 = T_2 + V_2$$

$$T_1 = 0 \qquad V_1 \cong W(R-r)\left(\theta_m^2/2\right)$$

$$T_2 = \tfrac{3}{4} m(R-r)^2 \dot{\theta}_m^2 \qquad V_2 = 0$$

$$0 + W(R-r)\frac{\theta_m^2}{2} = \tfrac{3}{4} m(R-r)^2 \dot{\theta}_m^2 + 0$$

$$(mg)(R-r)\frac{\theta_m^2}{2} = \tfrac{3}{4} m(R-r)^2 (\theta_m \omega_n)_m^2$$



$$\omega_n^2 = \frac{2}{3}\frac{g}{R-r}$$

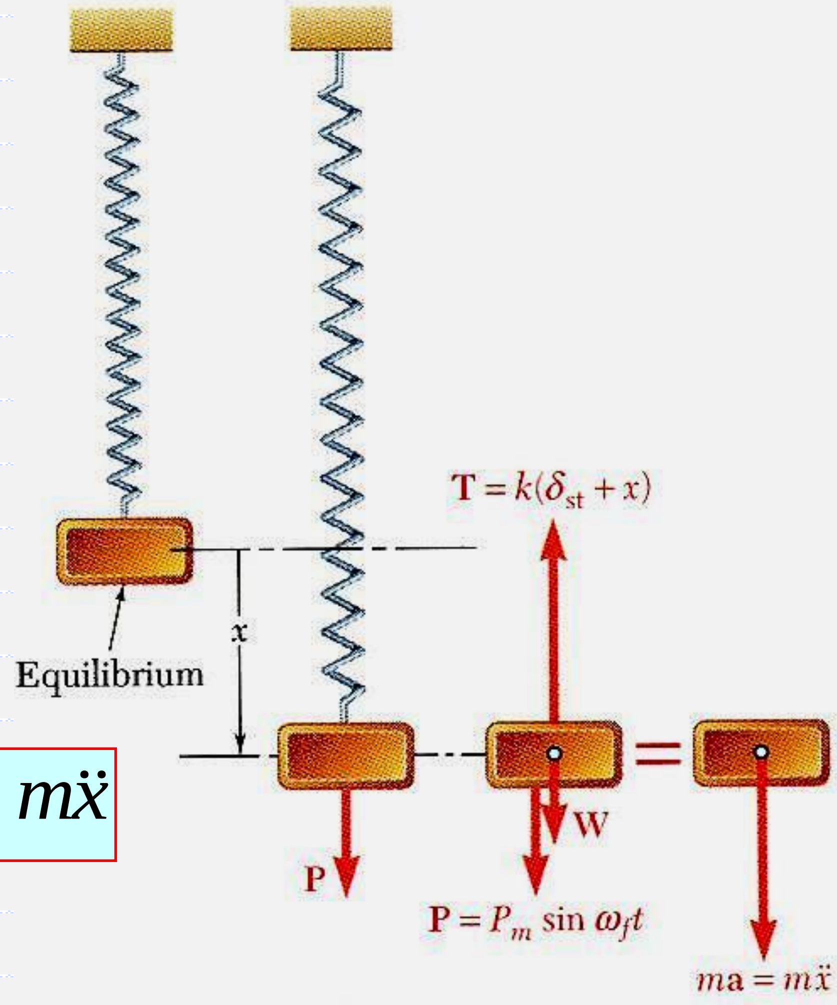
$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{3}{2}\frac{R-r}{g}}$$

Forced Vibrations

ارتعاش زوری (با نیروی خارجی)

- ***Forced vibrations*** **- Occur when a system is subjected to a periodic force or a periodic displacement of a support.**

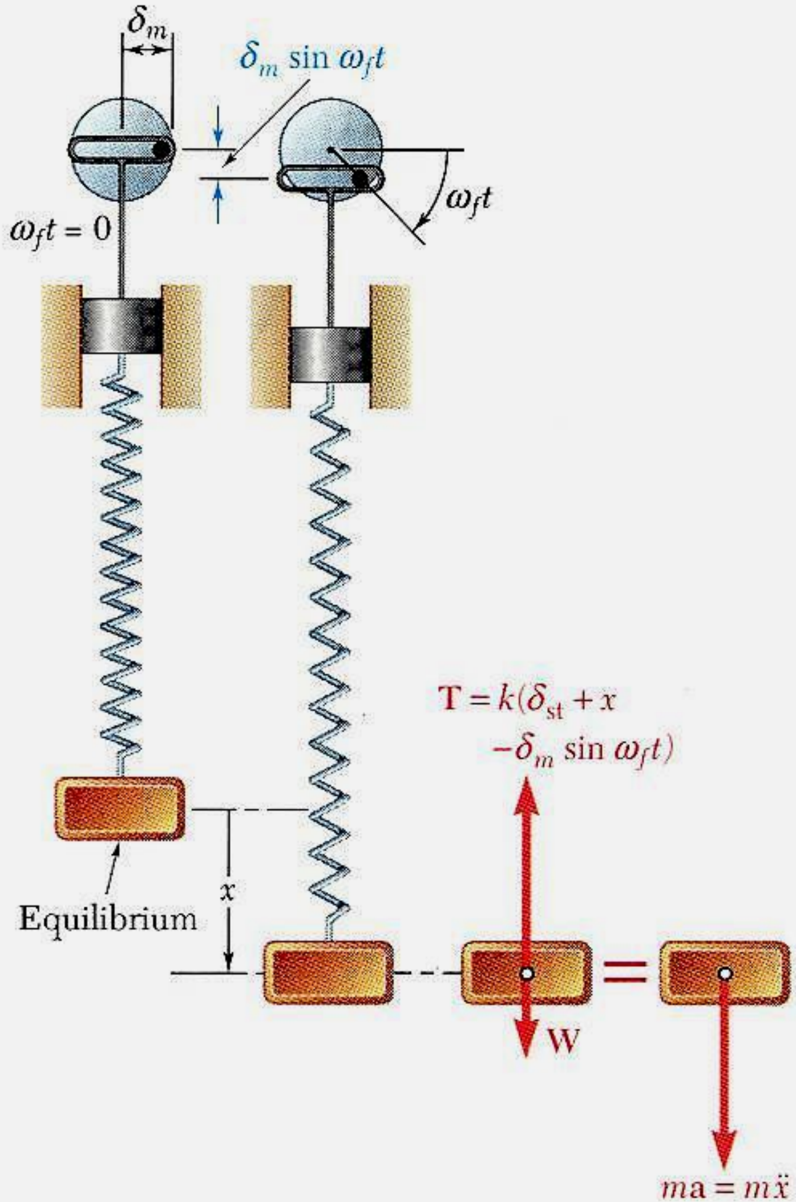
  ω_f = *forced frequency*



$$+\downarrow \sum F = ma:$$

$$P_m \sin \omega_f t + W - k(\delta_{st} + x) = m\ddot{x}$$

$$m\ddot{x} + kx = P_m \sin \omega_f t$$



ω_f = ***forced frequency***

$$+\downarrow \sum F = ma :$$

$$W - k\left(\delta_{st} + x - \delta_m \sin \omega_f t\right) = m\ddot{x}$$

$$m\ddot{x} + kx = k\delta_m \sin \omega_f t$$

$$m\ddot{x} + kx = P_m \sin \omega_f t$$

$$m\ddot{x} + kx = k\delta_m \sin \omega_f t$$

$$x = x_{comp.} + x_{part.}$$
$$= \left[A \sin \omega_n t + B \cos \omega_n t \right] + x_m \sin \omega_f t$$

Substituting particular solution into governing equation,

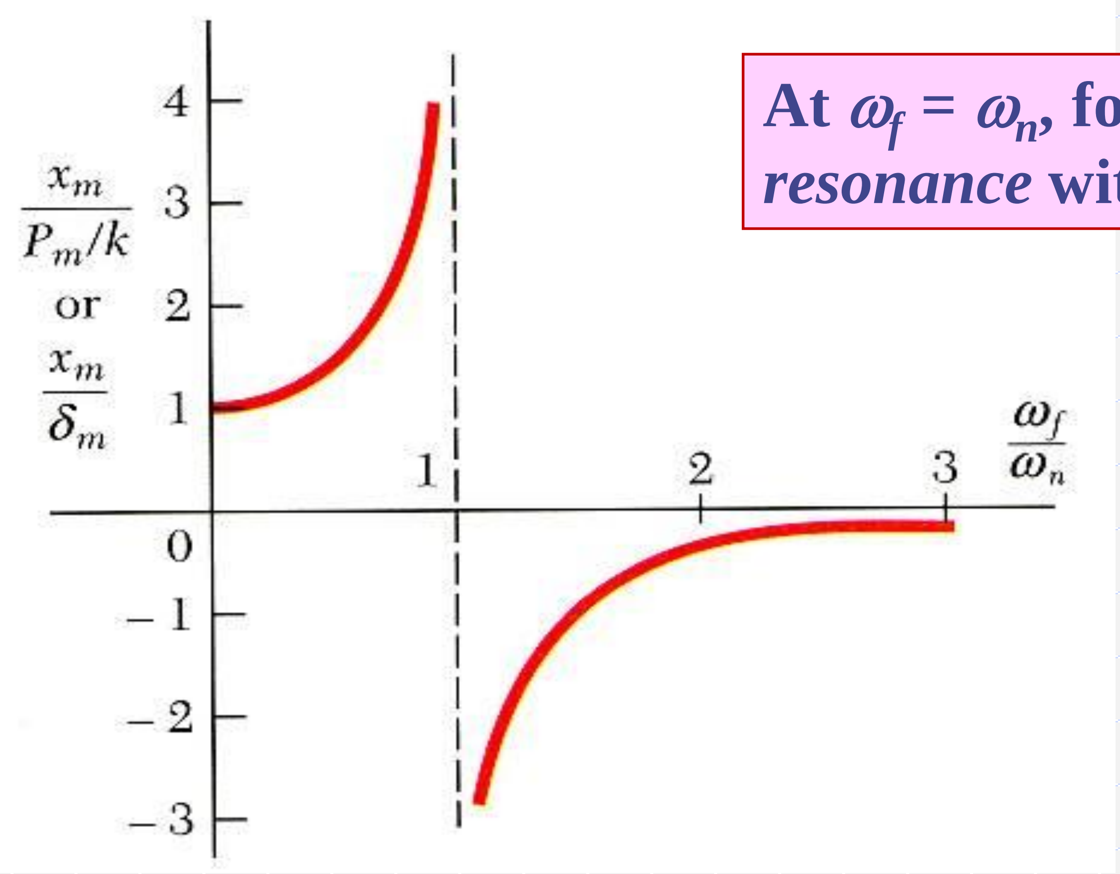
$$-m\omega_f^2 x_m \sin \omega_f t + kx_m \sin \omega_f t = P_m \sin \omega_f t$$

$$x_m = \frac{P_m}{k - m\omega_f^2} = \frac{P_m / k}{1 - (\omega_f / \omega_n)^2} = \frac{\delta_m}{1 - (\omega_f / \omega_n)^2}$$

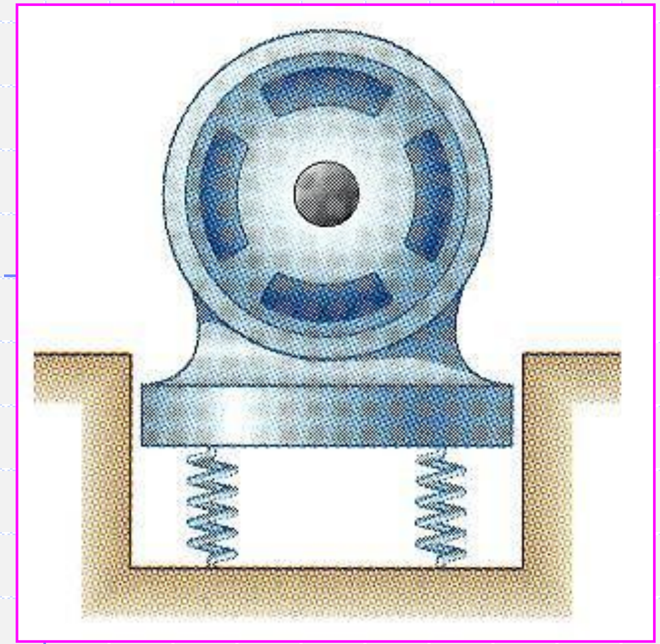
$$\text{Magnification factor} = \frac{x_m}{P_m / k} = \frac{x_m}{\delta_m} = \frac{1}{1 - (\omega_f / \omega_n)^2}$$

$$\text{Magnification factor} = \frac{x_m}{P_m/k} = \frac{x_m}{\delta_m}$$

$$= \frac{1}{1 - (\omega_f/\omega_n)^2}$$



At $\omega_f = \omega_n$, forcing input is in *resonance* with the system.

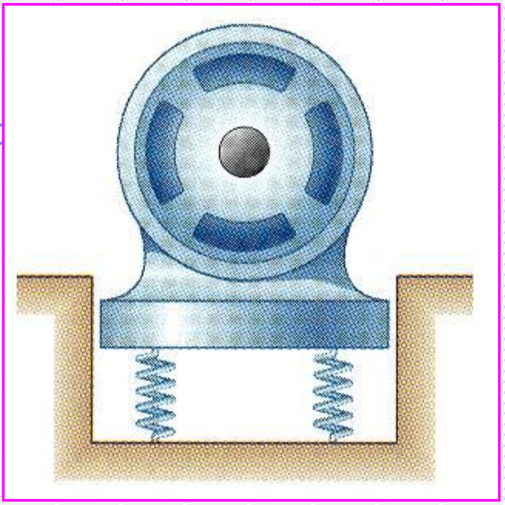


A motor weighing 350 lb is supported by four springs, each having a constant 750 lb/in. The unbalance of the motor is equivalent to a weight of 1 oz located 6 in. from the axis of rotation.

Determine *a*) speed in rpm at which resonance will occur, and *b*) amplitude of the vibration at 1200 rpm.

SOLUTION:

- The resonant frequency is equal to the natural frequency of the system.
- Evaluate the magnitude of the periodic force due to the motor unbalance. Determine the vibration amplitude from the frequency ratio at 1200 rpm.



W = 350 lb

k = 4(750 lb/in)

SOLUTION:

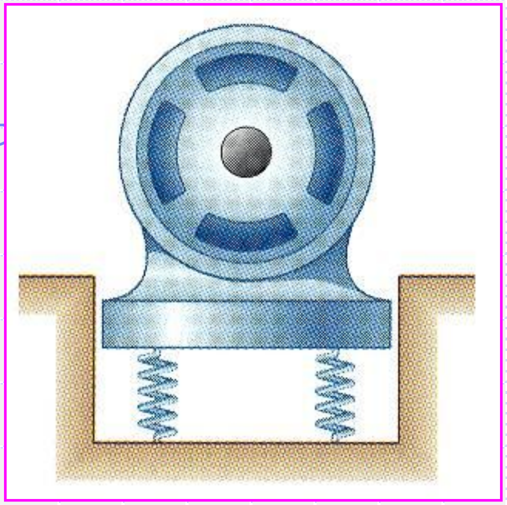
- The resonant frequency is equal to the natural frequency of the system.

$$m = \frac{350}{32.2} = 10.87\,\text{lb}\cdot\text{s}^2/\text{ft}$$

$$k = 4(750) = 3000\,\text{lb/in}$$
$$= 36{,}000\,\text{lb/ft}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{36{,}000}{10.87}}$$
$$= 57.5\,\text{rad/s} = 549\text{ rpm}$$

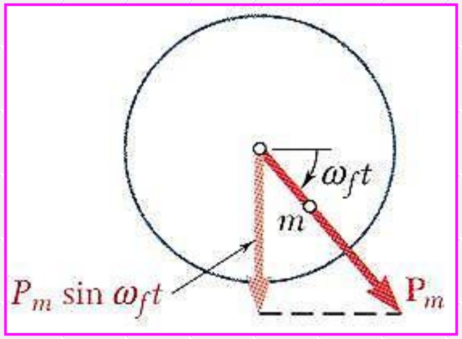
Resonance speed = 549 rpm



$W = 350$ lb

$k = 4(750$ lb/in)

$\omega_n = 57.5$ rad/s



- Evaluate the magnitude of the periodic force due to the motor unbalance. Determine the vibration amplitude from the frequency ratio at 1200 rpm.

$$\omega_f = \omega = 1200 \text{ rpm} = 125.7 \text{ rad/s}$$

$$m = (1 \text{ oz})\left(\frac{1 \text{ lb}}{16 \text{ oz}}\right)\left(\frac{1}{32.2 \text{ ft/s}^2}\right) = 0.001941 \text{ lb}\cdot\text{s}^2/\text{ft}$$

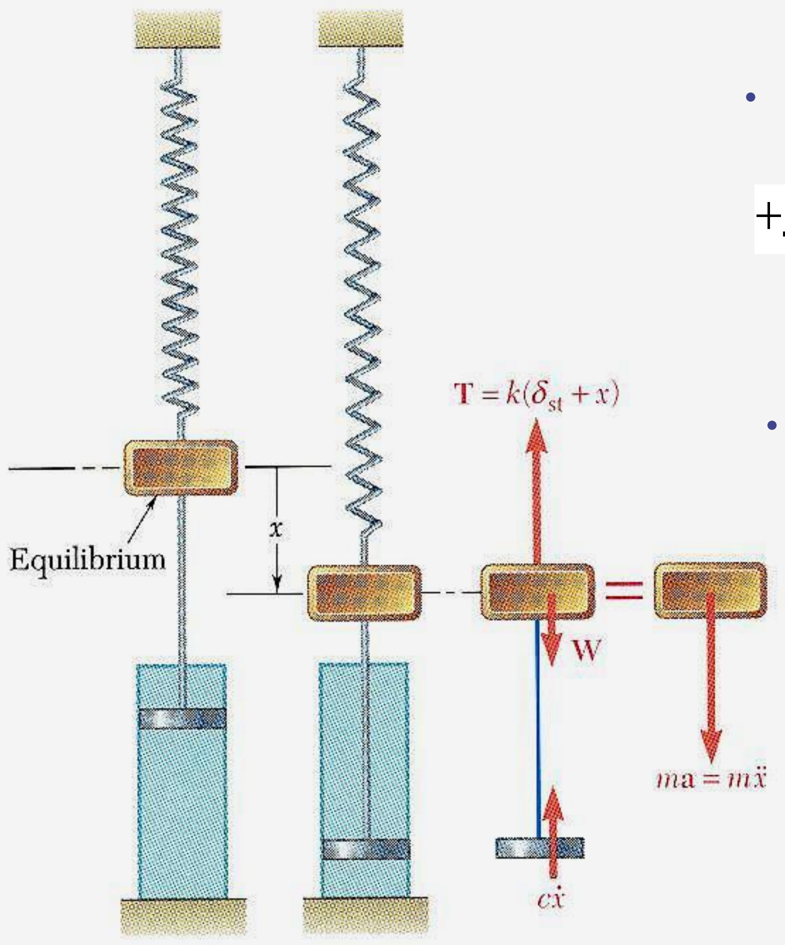
$$P_m = ma_n = mr\omega^2$$
$$= (0.001941)\left(\tfrac{6}{12}\right)(125.7)^2 = 15.33 \text{ lb}$$

$$x_m = \frac{P_m/k}{1-(\omega_f/\omega_n)^2} = \frac{15.33/3000}{1-(125.7/57.5)^2}$$
$$= -0.001352 \text{ in}$$

$x_m = 0.001352$ in. (out of phase)

Damped Free Vibrations

ارتعاش آزاد با میرائی



- **All vibrations are damped to some degree by forces due to *dry friction*, *fluid friction*, or *internal friction*.**

- **With viscous damping due to fluid friction,**

$$+\downarrow \sum F = ma: \quad W - k(\delta_{st} + x) - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

- ***Substituting $x = e^{\lambda t}$ and dividing through by $e^{\lambda t}$ yields the characteristic equation,***

$$m\lambda^2 + c\lambda + k = 0 \qquad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

- ***Define the critical damping coefficient such that***

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

- **Characteristic equation :**

$$m\lambda^2 + c\lambda + k = 0 \qquad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$c_c = 2m\omega_n =$ ***critical damping coefficient***

- ***Heavy damping :*** $c > c_c$

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

- negative roots
- nonvibratory motion

- ***Critical damping :*** $c = c_c$

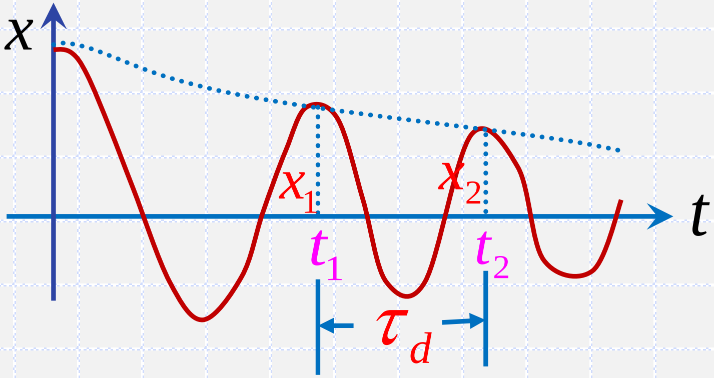
$$x = (A + Bt)e^{-\omega_n t}$$

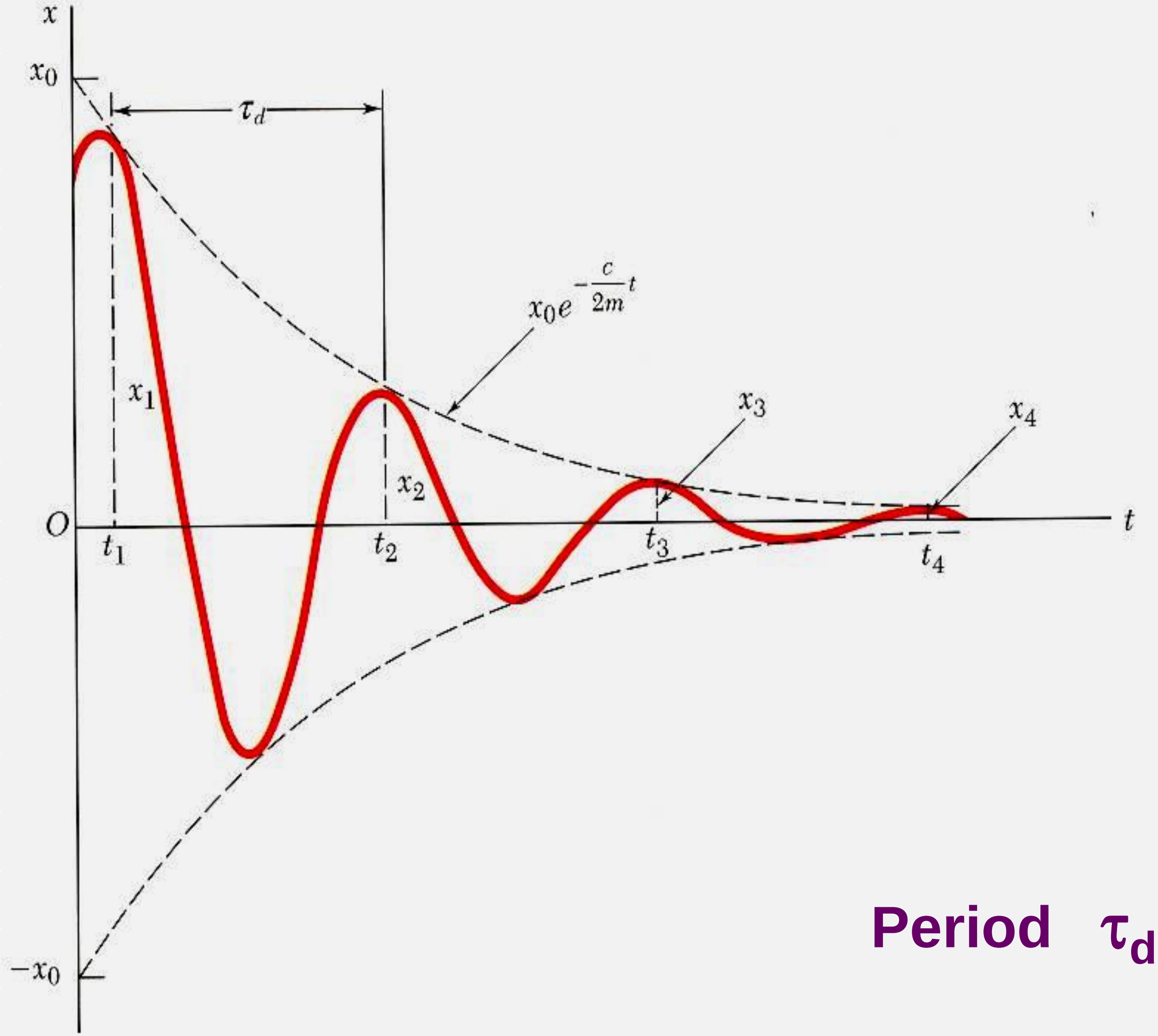
- double roots
- nonvibratory motion

- ***Light damping :*** $c < c_c$

$$x = e^{-(c/2m)t}\left(A\sin\omega_d t + B\cos\omega_d t\right)$$

$$\omega_d = \omega_n\sqrt{1 - \left(\frac{c}{c_c}\right)^2} =$$ ***damped frequency***

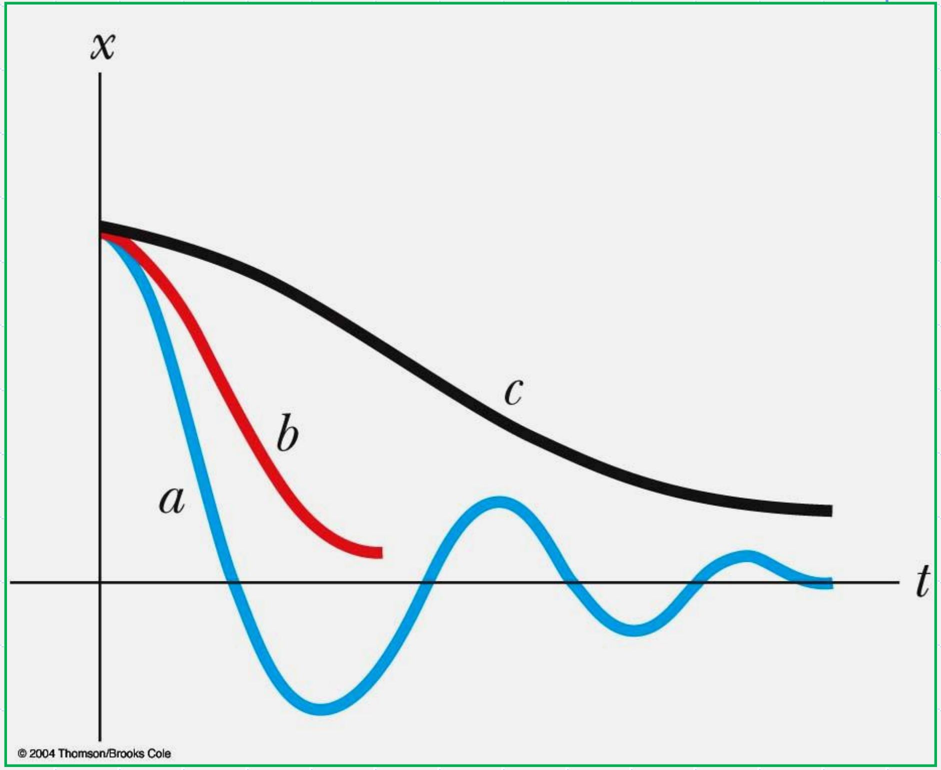




Period $\tau_d = 2\pi/\omega_d$

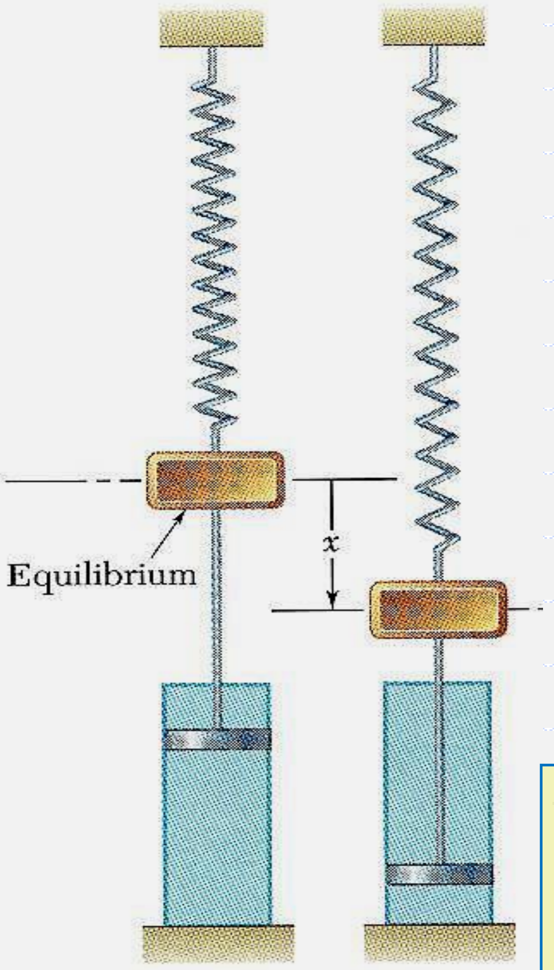
Types of Damping

- Graphs of position versus time for
  - (a) Light damped oscillator
  - (b) a critically damped oscillator
  - (c) Heavy damped oscillator
- For critically damped and heavy damped there is no angular frequency



Damped Forced Vibrations

ارتعاش زوری با میرائی



$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega_f t$$

$$x = x_{comp.} + x_{part.}$$

$$\frac{x_m}{P_m/k} = \frac{x_m}{\delta_m} = \frac{1}{\sqrt{[1 - (\omega_f/\omega_n)^2]^2 + [2(c/c_c)(\omega_f/\omega_n)]^2}}$$

$$\tan \varphi = \frac{2(c/c_c)(\omega_f/\omega_n)}{1 - (\omega_f/\omega_n)^2}$$

where $\omega_n = \sqrt{k/m}$ = natural circular frequency of undamped system

$c_c = 2m\,\omega_n$ = critical damping coefficient

c/c_c = damping factor

$$\frac{x_m}{P_m/k} = \frac{x_m}{\delta} = \frac{1}{\sqrt{\left[1-(\omega_f/\omega_n)^2\right]^2 + \left[2(c/c_c)(\omega_f/\omega_n)\right]^2}} =$$

magnification factor

$$\tan\phi = \frac{2(c/c_c)(\omega_f/\omega_n)}{1-(\omega_f/\omega_n)^2} =$$

phase difference between forcing and steady state response